Trace anomaly of dilaton coupled scalars in two dimensions

Raphael Bousso* and Stephen W. Hawking†

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street, Cambridge CB3 9EW

DAMTP/R-97/25 hep-th/9705236
submitted to Phys. Rev. Lett.

Abstract

Conformal scalar fields coupled to the dilaton appear naturally in two-dimensional models of black hole evaporation. We show that their trace anomaly is $(1/24\pi)[R - 6(\nabla\phi)^2 - 2\Box\phi]$. It follows that an RST-type counterterm appears naturally in the one-loop effective action.

* *R.Bousso@damtp.cam.ac.uk*
† *S.W.Hawking@damtp.cam.ac.uk*

1 Introduction

In the study of black hole radiation, many useful results have been obtained from two-dimensional (2D) models. It is hoped that the results will extend, at least partly, to the behaviour of realistic black holes in four or more dimensions. To make this claim plausible, the 2D actions were usually obtained by a dimensional reduction from a higher-dimensional theory. In the seminal papers of Callan, Giddings, Harvey, and Strominger (CGHS) [1], and of Russo, Susskind, and Thorlacius (RST) [2], the action is

$$S = -\frac{1}{16\pi G} \int d^2x \, \sqrt{-g} \, e^{-2\phi} \left(R + 4(\nabla\phi)^2 + 4\lambda^2 \right).$$

This action is argued by CGHS to describe the radial modes of extremal dilatonic black holes in four or more dimensions [3, 4]. Later, Trivedi and Strominger [5, 6] studied a 2D model that was obtained directly from 4D Einstein-Maxwell theory. A spherically symmetric ansatz,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{-2\phi} d\Omega^2,$$

allows the integration of the angular modes, and yields the action

$$S = -\frac{1}{16\pi G} \int d^2x \, \sqrt{-g} \, e^{-2\phi} \left(R + 2(\nabla\phi)^2 + 2e^{2\phi} - 2Q^2 e^{4\phi} \right).$$

For the description of black hole radiation, matter fields must be included in the theory. For conformal matter, the trace of the energy-momentum tensor vanishes classically; if treated as a quantum field, however, the trace acquires a non-zero expectation value on a curved background. The amount of radiation at infinity is directly proportional to the trace anomaly [7]. By including the trace anomaly in the equations of motion, or, equivalently, by including the one-loop effective action of the matter field, one can study the back reaction of the evaporation on the geometry.

The simplest kind of matter that might be used is a minimally coupled scalar field, f. To obtain a 2D model of genuine 4D pedigree, this field must be included in the 4D theory. It must be reduced to 2D like the rest of the action. Thus, the 4D classical action should be amended by a term proportional to

$$\int d^4x \, \left(-g^{(4)} \right)^{1/2} \left(\nabla^{(4)} f \right)^2 .$$

By reducing this term to 2D, one finds that the following matter contribution must be added to the 2D action:

$$S_{\mathrm{m}} = \frac{1}{2} \int d^2x \, \sqrt{-g} \, e^{-2\phi} (\nabla f)^2. \tag{1.1}$$

Thus, in the reduction process, the kinetic term acquires an exponential coupling to the dilaton.

The next step is to calculate the trace anomaly or the one-loop effective action, in order to include quantum effects. The field in Eq. (1.1) is conformally invariant; its trace anomaly, however, was not known. Because of this problem, CGHS included the field f as a minimally coupled field in 2D:

$$S_{\mathrm{m}} = \frac{1}{2} \int d^2x \, \sqrt{-g} \, (\nabla f)^2.$$

For this field the trace anomaly is known, but its action could not have arisen in the reduction from a realistic higher-dimensional theory. This inconsistency pervades the entire literature on 2D models. The problem was addressed most openly by Trivedi [5], who admitted that the 4D interpretation was lost when the minimal scalars are introduced into the 2D model. This interpretational gap seemed to become even wider when RST introduced a counterterm into the effective action by hand, rendering the model solvable but even less natural [2].

In this paper we calculate the trace anomaly for the dilaton coupled scalar field in two dimensions. This will make it possible to study black hole radiation in 2D models that have a genuine 4D interpretation. A particularly interesting result is that a counterterm of the same form as that introduced by RST appears naturally in the one-loop effective action for the dilaton coupled scalar. We will use the zeta function approach, together with general properties of the trace anomaly; a brief summary of these methods is given in the following section. A more extensive discussion is found, e.g., in Refs. [8, 9].

2 Methods

From the eigenvalues λ_n of the operator A, one defines a generalised zeta function,

$$\zeta(s) = \mathrm{tr} A^{-s} = \sum_n \lambda_n^{-s}.$$

This sum converges for a sufficiently large real part of s. By analytic extension, it defines a meromorphic function of s, which is regular even in regions where the sum diverges. The one-loop effective action, W, is given by

$$W = \frac{1}{2}\left[\zeta'(0) + \zeta(0)\log\mu^2\right], \tag{2.1}$$

where $\zeta' = d\zeta/ds$. Under a rescaling of the operator,

$$A[k] = k^{-1}A, \tag{2.2}$$

the one-loop effective action transforms as

$$W[k] = W + \frac{1}{2}\zeta(0)\log k. \tag{2.3}$$

We denote the trace anomaly by T. Let us summarise some of its general properties in D-dimensional spacetimes [8]. If D is odd, the trace anomaly is zero. If D is even, it consists of terms T_i that are generally covariant and homogeneous of order D in derivatives: $T = \sum q_i T_i$. The dimensionless numbers q_i are universal, i.e., independent of the background metric. This property will be particularly useful, because it will allow us to choose convenient backgrounds to determine the values of the q_i.

The integral of the trace anomaly over the manifold is given by:

$$\int d^D x \sqrt{g}\, T = 2\left.\frac{dW}{dk}\right|_{k=1}, \tag{2.4}$$

where k is defined as a scale factor of the metric, $\hat{g}^{\mu\nu} = k^{-1}g^{\mu\nu}$. But under this scale transformation, the eigenvalues of A transform as in Eq. (2.2). Therefore, Eq. (2.3) can be used, which yields the elegant result

$$\int d^D x \sqrt{g}\, T = \zeta(0). \tag{2.5}$$

Given an operator A, one could, in principle, calculate the one-loop effective action directly from Eq. (2.1). In practice, it is often simpler to calculate the trace anomaly from Eq. (2.5), because the zeta function is usually easier to obtain than its derivative. By requiring that Eq. (2.4) hold, the effective action can be inferred up to terms that do not depend on the scale factor. (We shall use W^* to denote a quantity which differs from the effective action only by such terms.) Also, if the effective action is known for the operator kA, one can use Eq. (2.3) to obtain W for the operator A.

3 Dilaton Coupled Scalar

By Eq. (1.1), scalar fields obtained through dimensional reduction from four dimensions will have an $e^{-2\phi}$ dilaton coupling in the action. Variation with respect to f yields the equation of motion $Af = 0$, with the field operator

$$A = e^{-2\phi} \left(-\Box + 2\nabla^\mu \phi \nabla_\mu\right). \tag{3.1}$$

The trace anomaly consists of covariant terms with two metric derivatives. For the operator at hand, there are only three such expressions: R, $(\nabla\phi)^2$, and $\Box\phi$. In principle, these terms could still be multiplied by arbitrary functions of ϕ. But consider shifting ϕ by a constant value $\Delta\phi$. This corresponds merely to multiplying the kinetic term in the action by a factor $e^{-2\Delta\phi}$; the trace anomaly will remain the same. Therefore a functional dependence of any of its terms on ϕ can be excluded. Consequently, we can write

$$T = q_1 R + q_2 (\nabla\phi)^2 + q_3 \Box\phi. \tag{3.2}$$

By writing the metric in conformal gauge,

$$ds^2 = e^{2\rho(t,x)} \left(dt^2 + dx^2\right),$$

it is easy to check that this anomaly derives from the effective action

$$W^* = -\frac{1}{2} \int d^2x \sqrt{g} \left[\frac{q_1}{2} R \frac{1}{\Box} R + q_2 (\nabla\phi)^2 \frac{1}{\Box} R + q_3 \phi R\right]. \tag{3.3}$$

This follows from Eq. (2.4), since $R = -2\Box\rho$. (A more straightforward result for the the last term would be $\Box\phi \frac{1}{\Box} R$. It is related to the term we use by two integrations by parts; the difference can at most be a boundary term. It will become clear below why we choose the form ϕR.) We must only determine the universal numbers q_1, q_2, and q_3 to obtain the trace anomaly completely.

3.1 Coefficients of R and of $\Box\phi$

First consider the case when ϕ is identically zero. Then Eq. (3.2) simplifies to $T_{\phi\equiv 0} = q_1 R$. But if $\phi \equiv 0$, the operator A in Eq. (3.1) becomes $A_{\phi\equiv 0} = -\Box$. This is the operator for the minimally coupled scalar, for which the trace anomaly is well known [7]: $T_{\min} = R/24\pi$. Therefore, one finds that

$$q_1 = \frac{1}{24\pi}.$$

Now consider the case where ϕ is constant, $\phi \equiv \phi_c$. Then the one-loop effective action, Eq. (3.3), simplifies to

$$W^*_{\phi\equiv\phi_c} = W^*_{\min} - \frac{1}{2}\int d^2x\sqrt{g}\, q_3\phi_c R. \tag{3.4}$$

To make sure that the integral over the Ricci scalar does not vanish, we can specify that a background with the topology of a two-sphere be used. For constant ϕ, the operator A becomes $A_{\phi\equiv\phi_c} = -e^{-2\phi_c}\Box$. But this is just the minimally coupled operator, rescaled by a constant factor $k^{-1} = e^{-2\phi_c}$. Therefore, Eqs. (2.3) and (2.5) yield:

$$\begin{aligned} W^*_{\phi\equiv\phi_c} &= W^*_{\min} + \phi_c\zeta_{\min}(0) \\ &= W^*_{\min} + \phi_c q_1 \int d^2x\sqrt{g}\, R. \end{aligned} \tag{3.5}$$

Comparison with Eq. (3.4) shows that

$$q_3 = -2q_1 = -\frac{1}{12\pi}.$$

The same consideration also vindicates the choice of ϕR for the last term in the effective action, Eq. (3.3): If $\Box\phi\frac{1}{\Box}R$ was used, the last term in Eq. (3.4) would be zero, since ϕ is constant. It would then be impossible to match Eq. (3.4) to Eq. (3.5), in which the last term is non-zero on a two-sphere background.[1]

3.2 Coefficient of $(\nabla\phi)^2$

In conformal gauge the field operator will take the form

$$A = e^{-2\phi-2\rho}\left[-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 2\left(\frac{\partial\phi}{\partial t}\frac{\partial}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\right)\right].$$

Consider a Euclidean background manifold of toroidal topology, in which t and x are periodically identified, with period 2π. The integral over the Ricci scalar is a topological invariant and vanishes on a torus. Since $\Box\phi$ is a total divergence, its integral vanishes as well. Thus,

$$\zeta(0) = \int d^2x\sqrt{g}\, T = q_2\int d^2x\sqrt{g}\,(\nabla\phi)^2. \tag{3.6}$$

[1] Nojiri and Odintsov [10] suggest a more general form for the effective action, in which the last term is given by $q_3[a\phi R + (1-a)\Box\phi\frac{1}{\Box}R]$. This would give a different value of q_3.

Therefore we can determine q_2 by calculating $\zeta(0)$ from the operator eigenvalues in a conveniently chosen toroidal background, and dividing the result by $\int d^2x\sqrt{g}\,(\nabla\phi)^2$.

A useful choice of background is the field configuration $\phi = -\rho = \epsilon \sin t$, where $\epsilon \ll 1$. The operator takes the form

$$A = -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 2\epsilon \cos t \frac{\partial}{\partial t}.$$

For $\epsilon = 0$, this operator is just the flat space Laplacian, for which $\zeta(0)$ is known to vanish. The integral on the right hand side of Eq. (3.6) yields $2\pi^2\epsilon^2$. Thus we can proceed as follows: The eigenvalues of A will be found perturbatively in ϵ. This will allow us to expand $\zeta(s)$ to second order in ϵ:

$$\zeta(s) = \zeta^{(0)}(s) + \epsilon\zeta^{(1)}(s) + \epsilon^2\zeta^{(2)}(s).$$

Since $\zeta^{(0)}(0) = 0$, we have

$$\epsilon\zeta^{(1)}(0) + \epsilon^2\zeta^{(2)}(0) = 2\pi^2 q_2 \epsilon^2.$$

Consistency requires that $\zeta^{(1)}(0) = 0$; we will indeed find that to be the case. Therefore,

$$q_2 = \frac{1}{2\pi^2}\zeta^{(2)}(0). \tag{3.7}$$

For $\epsilon = 0$, the eigenvalues of the operator A are $\Lambda^{(0)}_{kl} = k^2 + l^2$, with degeneracies

$$d(k,l) = \begin{cases} 4 & \text{if } k \geq 1,\ l \geq 1 \\ 2 & \text{if } k \geq 1,\ l = 0 \text{ or } k = 0,\ l \geq 1 \\ 1 & \text{if } k = l = 0. \end{cases}$$

Clearly, the zeta function,

$$\zeta(s) = \sum_{k,\,l=0}^{\infty} d(k,l)\left(\Lambda^{(0)}_{kl}\right)^{-s},$$

contains an ill-defined term: $k = l = 0$. This problem can be dealt with by introducing a mass term into the operator A: $A \to A + M^2$. Then $\zeta(0)$ can be defined in the limit as $M \to 0$.

Now take $\epsilon \neq 0$, and consider the eigenvalue equation, $Af = \Lambda f$. With $f(t,x) = T(t)X(x)$ the equation separates into

$$-X'' = \sigma X, \quad -\ddot{T} + 2\epsilon \cos t\, \dot{T} = \lambda T.$$

Standard perturbation theory yields that, to second order in ϵ, the eigenvalues of A are

$$\Lambda_{kl} = k^2 + l^2 + M^2 + \epsilon^2 \frac{2l^2}{4l^2 - 1},$$

with the same degeneracies $d(k,l)$ as in the unperturbed case. The zeta function is given by

$$\begin{aligned}
\zeta(s) &= \sum_{k,\, l=0}^{\infty} d(k,l) \left(\Lambda_{kl}^{(0)}\right)^{-s} \left(1 + \epsilon^2 \frac{\lambda_l^{(2)}}{\Lambda_{kl}^{(0)}}\right)^{-s} \\
&= \sum_{k,\, l=0}^{\infty} d(k,l) \left(\Lambda_{kl}^{(0)}\right)^{-s} \left(1 - \epsilon^2 s \frac{\lambda_l^{(2)}}{\Lambda_{kl}^{(0)}}\right) \\
&= \zeta^{(0)}(s) - \epsilon^2 s \sum_{k=0,\, l=1}^{\infty} d(k,l) \frac{\lambda_l^{(2)}}{\left(\Lambda_{kl}^{(0)}\right)^{1+s}},
\end{aligned}$$

where a Taylor expansion to second order in ϵ was used. The sum in the last line does not include $l = 0$ because $\lambda_0^{(2)} = 0$. Since this excludes $k = l = 0$, it is safe to drop M at this point. Thus we have

$$\zeta^{(2)}(0) = -\lim_{s \to 0} sU(s), \tag{3.8}$$

where we view the double sum as a meromorphic function of s,

$$U(s) = \sum_{k=0,\, l=1}^{\infty} d(k,l) \frac{2l^2}{(k^2 + l^2)^{1+s}(4l^2 - 1)}.$$

If $U(0)$ were finite, $\zeta^{(2)}(0)$ would vanish; but it is easy to check that this is not the case. If U had poles of order 2 or greater, $\zeta^{(2)}(0)$ would diverge. Only if U has a simple pole at $s = 0$ will we obtain a non-zero, finite result for $\zeta^{(2)}(0)$, and thus for q_2. We show below that this is indeed the case.

To understand fully the behaviour of $U(s)$, we would have to evaluate it in terms of known meromorphic functions. Fortunately we need to find only the principal part of the Laurent series of U around $s = 0$, $\mathrm{Pr}[U(s); 0]$,

because the regular part will be annulled by the factor of s in Eq. (3.8). But $\Pr[U(s);0] = \Pr[U(s)+V(s);0]$ for any function V(s) which is regular at $s=0$. Thus, by adding suitable finite terms to the double sum, we can bring it into a form which can be evaluated.

First, we note that the contribution from $k=0$ is finite at $s=0$:

$$2\sum_{l=1}^{\infty}\frac{2}{4l^2-1} = 2.$$

After its subtraction, all summations start at 1:

$$\Pr[U(s);0] = 2\Pr\left[\sum_{k,\,l=1}^{\infty}\frac{4l^2}{(k^2+l^2)^{1+s}(4l^2-1)};0\right],$$

where we have used $d(k,l)=4$. Next, we subtract 1 in the numerator; this is possible since $\sum(k^2+l^2)^{-1-s}(4l^2-1)^{-1}$ is finite at $s=0$ (the upper bound $-1+\frac{\pi^2}{12}+(\log 2-\frac{1}{2})\pi\coth\pi \approx 0.43$ is easily found). The cancellation of terms yields

$$\Pr[U(s);0] = 2\Pr\left[\sum_{k,\,l=1}^{\infty}\frac{1}{(k^2+l^2)^{1+s}};0\right].$$

But

$$\sum_{k,\,l=1}^{\infty}\frac{1}{(k^2+l^2)^{1+s}} = \frac{1}{4}Z_2(2+2s)-\zeta_{\mathrm{R}}(2+2s),$$

where

$$Z_2(p) = \sum_{k,\,l=-\infty}^{\infty}{}' \,(k^2+l^2)^{-p/2}$$

is a generalised zeta function of Epstein type; the prime denotes the omission of the $k=l=0$ term in the sum. Epstein showed in Ref. [11] that $Z_2(p)$ is analytic except for a simple pole at $p=2$, with residue 2π. Since the Riemann zeta function $\zeta_{\mathrm{R}}(2+2s)$ is finite for $s=0$, we find

$$2\Pr[U(s);0] = \frac{1}{2}\Pr\left[Z_2(2+2s);0\right] = \frac{1}{2}\left(\frac{2\pi}{2s}\right) = \frac{\pi}{2s}.$$

Therefore, by Eq. (3.8), we find that $\zeta^{(2)}(0) = -\pi/2$, and, by Eq. (3.7), we obtain the result

$$q_2 = -\frac{1}{4\pi}.$$

4 Summary

We have shown that a 2D conformal scalar field with exponential dilaton coupling has the trace anomaly

$$T = \frac{1}{24\pi} \left[R - 6(\nabla\phi)^2 - 2\Box\phi \right].$$

The scale factor dependent part of the one-loop effective action is

$$W^* = -\frac{1}{48\pi} \int d^2x \sqrt{g} \left[\frac{1}{2} R \frac{1}{\Box} R - 6(\nabla\phi)^2 \frac{1}{\Box} R - 2\phi R \right].$$

This is the proper one-loop term that should be used in 2D investigations of black hole evaporation. It is interesting to note that the last term was inserted by hand in the RST model, albeit with a different coefficient. By Eq. (2.1), the effective action will also contain a term $(1/2)\zeta(0)\log\mu^2$. The R and $\Box\phi$ terms in the trace anomaly give only a topological contribution to $\zeta(0)$, which does not affect the equations of motion. The term

$$-(1/8\pi)\log\mu^2 \int d^2x \sqrt{g}\, (\nabla\phi)^2,$$

however, must be taken into account.

The minimally coupled scalars, which are usually included in the 2D theory to carry the black hole radiation, have no higher-dimensional interpretation. With our result, it will be possible to investigate, for the first time, 2D models derived entirely from higher-dimensional theories.

Acknowledgements

We thank G. Gibbons and M. Perry for discussions. We are grateful to S. Nojiri, S. Odintsov, D. Vassilevich, and E. Elizalde for their comments on an earlier version of this letter.

References

[1] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger: *Evanescent black holes*. Phys. Rev. D **45**, 1005 (1992), hep-th/9111056.

[2] J. G. Russo, L. Susskind and L. Thorlacius: *Black hole evaporation in $1+1$ dimensions*. Phys. Lett. **B292**, 13 (1992), hep-th/9201074.

[3] S. B. Giddings and A. Strominger: *Exact black fivebranes in critical superstring theory*. Phys. Rev. Lett. **67**, 2930 (1991).

[4] D. Garfinkle, G. T. Horowitz and A. Strominger: *Charged black holes in string theory*. Phys. Rev. D **43**, 3140 (1991).

[5] S. P. Trivedi: *Semiclassical extremal black holes*. Phys. Rev. D **47**, 4233 (1993), hep-th/9211011.

[6] A. Strominger and S. P. Trivedi: *Information consumption by Reissner-Nordström black holes*. Phys. Rev. D **48**, 5778 (1993), hep-th/9302080.

[7] S. M. Christensen and S. A. Fulling: *Trace anomalies and the Hawking effect*. Phys. Rev. D **15**, 2088 (1977).

[8] N. D. Birrell and P. C. W. Davies: *Quantum fields in curved space*. Cambridge University Press, Cambridge, England (1982).

[9] S. W. Hawking: *Zeta function regularization of path integrals in curved spacetime*. Commun. math. Phys. **55**, 133 (1977).

[10] S. Nojiri and S. D. Odintsov: *Trace anomaly and non-local effective action for 2D conformally invariant scalar interacting with dilaton*. E-print no. hep-th/9706009.

[11] P. Epstein: *Zur Theorie allgemeiner Zetafunctionen*. Mathematische Annalen **56**, 615 (1903).